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Rational Lax operators and their quantization

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Abstract

We investigate the construction of the quantum commuting hamiltonians for the Gaudin integrable model. We prove that $[TrL^k(z), TrL^m(u)] = 0$, for $k, m < 4$. However this naive receipt of quantization of classically commuting hamiltonians fails in general, for example we prove that $[TrL^4(z), TrL^2(u)] \neq 0$. We investigate in details the case of the one spin Gaudin model with the magnetic field also known as the model obtained by the "argument shift method". Mathematically speaking this method gives maximal Poisson commutative subalgebras in the symmetric algebra $S(\mathfrak{gl}(N))$. We show that such subalgebras can be lifted to $U(\mathfrak{gl}(N))$, simply considering $TrL(z)^k$, $k \leq N$ for $N < 5$. For $N = 6$ this method fails: $[TrL_{MF}(z)^6, L_{MF}(u)^3] \neq 0$. All the proofs are based on the explicit calculations using r-matrix technique. We also propose the general receipt to find the commutation formula for powers of Lax operator. For small power exponents we find the complete commutation relations between powers of Lax operators.

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Contents

1 Introduction

This paper is devoted to the investigation of rational Lax operators of Gaudin type on quantum the level. The Gaudin model was introduced by Gaudin in [1] (see section 13.2.2) as a limit of the famous XXX-Heisenberg model, which describes interaction of spins in one dimensional chain. It appears to be that Gaudin model related to various fields of research in mathematics and mathematical physics: Knizhnik-Zamolodchikov equation [2] and isomonodromy deformation theory (see [3]); Hitchin system [4] (see [5], also [6, 7]); Langlands correspondence (see [8]); geometry of polygons (see [9, 10]). Also it seems to be not so widely known that integrable systems obtained by argument shift method [11], are the most simplest particular (one spin) cases of Gaudin model, more precisely one should add to the Gaudin Lax operator constant matrix, (physically this means turning on the magnetic field).

But despite lots of results concerning the Gaudin model it seems that some simple questions remain still open. One of such questions is explicit construction of higher Gaudin's hamiltonians on the quantum level. On the classical level the Gaudin hamiltonians can be obtained as values at different z of $TrL^k(z)$, where $L(z)$ is the Lax operator (see below). They are not independent but the it can be chosen a basis in this family of functions composed by coefficients of expansions at poles. We investigate the question whether the same construction on the quantum level gives commuting hamiltonians, we prove that $[TrL^k(z), TrL^m(u)] = 0$ for $k \leq 3, m \leq 3, \forall z, u$, but $[TrL^4(z), TrL^2(u)] \neq 0$. So by considerations of $TrL^k(z)$ one can find enough commuting hamiltonians for $\mathfrak{gl}(2), \mathfrak{gl}(3)$, but not for the higher rank. We also find some general formulas for the commutation between

matrix elements of powers $(L^k(z))_{i,j}$. The special case of the one spin Gaudin model (equivalent to the model constructed by the argument shift method) is more optimistic, we obtain: $[TrL^k(z), TrL^2(u)] = 0$ for $\forall k, z, u$, $[TrL^k(z), TrL^3(u)] = 0$ for $k \leq 5 \forall z, u$, but $[TrL^6(z), TrL^3(u)] \neq 0$, which means that in this particular case one is also unable to quantize the Poisson commuting subalgebra by considering the $TrL^k(z)$, starting from $\mathfrak{gl}(6)$.

Let us mention that the problem of lift of Poisson commutative subalgebra generated by the argument shift method has been investigated before: in [12] the method of quantization was proposed on the basis of Yangian technique, another receipt was proposed in [13], and it's also worth to mention that in [14] the Vinberg's conjecture (see [15]) that the symmetrization map from $S(\mathfrak{gl}(n)) \to U(\mathfrak{gl}(n))$ provides the necessary quantization was proved. Despite all these results it seems that the solution of the problem is not in the stage as one can hope: one can hope for some simple formula like $TrL^k(z)$ or some modification of it for the quantization of such subalgebras.

The same can be said about the Gaudin model: in [8, 16, 17] it was shown that the Gaudin hamiltonians can be obtained from the center on the critical level of the universal enveloping of the corresponding Kac-Moody algebra. It was proved in [16] that the center on the critical level is big enough, but there is no explicit construction of the center. On the other hand one can try to obtain the higher Gaudin hamiltonians from the known commuting hamiltonians of the XXX-Heisenberg spin chain.

Let us present the result of our paper in more explicit way. Let us recall some notations first: let us denote by Φ the following $n \times n$-matrix with coefficients in $\mathfrak{gl}(n)$: we put element e_{ij} on ij-th place of the matrix. We also consider the direct sum $\mathfrak{gl}(n) \oplus ... \oplus \mathfrak{gl}(n)$ and denote by Φ_i the matrix defined as above but with the elements from the i-th copy of $\mathfrak{gl}(n)$ in $\mathfrak{gl}(n) \oplus ... \oplus \mathfrak{gl}(n)$. Let us introduce the Lax operator for the Gaudin model:

$$L(z) = \mathcal{K} + \sum_{i=1...N} \frac{\Phi_i}{z - a_i} \tag{1}$$

where $a_i \in \mathbb{C}, \Phi_i...., \mathcal{K}$ arbitrary constant matrix, (physically $\mathcal{K}$ corresponds to a magnetic field.)

$$TrL^2(z) = \sum_i \frac{Tr\Phi_i^2}{(z - a_i)^2} + \sum_k \frac{1}{z - a_k}(Tr\mathcal{K}\Phi_k + \sum_{j \neq k} \frac{2Tr\Phi_k\Phi_j}{(a_k - a_j)}) \tag{2}$$

$$H_k = Tr\mathcal{K}\Phi_k + \sum_{j \neq k} \frac{2Tr\Phi_k\Phi_j}{(a_k - a_j)} \in U(\mathfrak{gl}(n)) \otimes ... \otimes U(\mathfrak{gl}(n)) \tag{3}$$

- are called quadratic Gaudin Hamiltonians

It's known that $[TrL^2(u), TrL^2(z)] = 0$ and so that $[H_k, H_j] = 0$. The Gaudin model on quantum level consists of taking some representation $V_1 \otimes ... \otimes V_N$ of $\mathfrak{gl}(n) \oplus ... \oplus \mathfrak{gl}(n)$ and asking for the spectrum of such operators, their matrix elements etc.

On the classical level one uses the symbol map $\mathsf{symb} : U(\mathfrak{gl}(n)) \to S(\mathfrak{gl}(n))$ and so the images of H_i are functions on the $\mathfrak{gl}(n)^* \oplus ... \oplus \mathfrak{gl}(n)^*$ so one can restrict them to the orbits

$\mathcal{O}_1 \times ... \times \mathcal{O}_N$, and so one obtains the phase space $\mathcal{O}_1 \times ... \times \mathcal{O}_N$ with Kirillov's symplectic form and functions $\mathsf{symb}(H_i)$ each of which can be taken as hamiltonian, so the classical hamiltonian systems is described. It's known that it is completely integrable: one can take coefficients at $(z-a_i)^{-m}$ of $\mathsf{symb}(TrL^k(z))$ as hamiltonians which will Poisson commute and (at least for the general choice of orbits, it seems that the question of considering different non-general orbits has not been analyzed carefully) this will give the number of independent hamiltonians equal to the half of the dimension of the phase space which means the complete integrability in the Liouville sense.

The fact that $Tr(L^k(z))$ and $Tr(L^l(u))$ commute with each other for any z, u, k, l with respect to the Poisson bracket can be proved by the r-matrix technique on the basis of commutation relation between $(L^k(z))_{kl}, (L^l(u))_{qp}$. Leningrad's notation seems to be very useful for such calculation: we denote by $\overset{1}{T} = T \otimes Id, \overset{2}{T} = Id \otimes T$. So $[\overset{1}{A}, \overset{2}{B}]_{ij,kl} = [a_{ij}, b_{kl}]$, hence the commutation relation between $\overset{1}{A}$ and $\overset{2}{B}$ encodes all the commutation relations between all a_{ij} and b_{kl}. Matrix P is defined by the rule: $P(a \otimes b) = b \otimes a$.

Our first results are the following formulas for the commutation relation between powers of Lax operator $L(z)$:

$$\begin{aligned} [\overset{1}{L}{}^n(z), \overset{2}{L}(u)] &= \overset{2}{L}{}^n(z)\frac{P}{z-u} - \overset{1}{L}{}^n(z)\frac{P}{z-u} + \\ &\sum_{i=0}^{n-1} \overset{1}{L}{}^i(z)(\overset{1}{L}(u) - \overset{2}{L}(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u} \end{aligned} \tag{4}$$

$$\begin{aligned} [\overset{1}{L}{}^n(z), \overset{2}{L}{}^2(u)] &= (\overset{1}{L}(u) + \overset{2}{L}(u))\overset{2}{L}{}^n(z)\frac{P}{z-u} - \overset{1}{L}{}^n(z)(\overset{1}{L}(u) + \overset{2}{L}(u))\frac{P}{z-u} + \\ &+ \sum_{i=0}^{n-1} \overset{1}{L}{}^i(z)(\overset{1}{L}{}^2(u) - \overset{2}{L}{}^2(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u} \end{aligned} \tag{5}$$

$$\begin{aligned} [\overset{1}{L}{}^n(z), \overset{2}{L}{}^3(u)] &= \sum_{i=0}^{n-1} \overset{1}{L}{}^i(z)(\overset{1}{L}{}^3(u) - \overset{2}{L}{}^3(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u} + \\ &+ (\overset{1}{L}{}^2(u) + \overset{1}{L}(u)\overset{2}{L}(u) + \overset{2}{L}{}^2(u))\overset{2}{L}{}^n(z)\frac{P}{z-u} \\ &- \overset{1}{L}{}^n(z)(\overset{1}{L}{}^2(u) + \overset{1}{L}(u)\overset{2}{L}(u) + \overset{2}{L}{}^2(u))\frac{P}{z-u} + \\ &+ [\frac{\partial}{\partial u}\overset{1}{L}(u) + \frac{\overset{1}{L}(u)}{z-u}, \overset{1}{L}{}^n(z)]\frac{1}{z-u} \end{aligned} \tag{6}$$

$$
\begin{aligned}
[\overset{1}{L}{}^n(z),\overset{2}{L}{}^m(u)] &= \sum_{i=0}^{n-1}\overset{1}{L}{}^i(z)(\overset{1}{L}{}^m(u)-\overset{2}{L}{}^m(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u}+\\
&+(\sum_{i=0}^{m-1}\overset{1}{L}{}^i(u)\overset{2}{L}{}^{m-1-i}(u))\overset{2}{L}{}^n(z)\frac{P}{z-u}-\\
&-\overset{1}{L}{}^n(z)(\sum_{i=0}^{m-1}\overset{1}{L}{}^i(u)\overset{2}{L}{}^{m-1-i}(u))\frac{P}{z-u}+\\
&+\text{ terms of lower degree} \qquad (7)
\end{aligned}
$$

Remark 1 *Let us emphasize the importance of such formulas. The special kind of normal ordering adapted to calculation of traces is realized therein. Indeed, in the right hand side we see the sums of terms of the type* $\overset{1}{A}_1\overset{1}{A}_2...\overset{1}{A}_n\overset{2}{B}_1\overset{2}{B}_2...\overset{2}{B}_lP$, *and there are no terms of the type:* $Tr\overset{1}{A}\overset{2}{B}\overset{1}{C}\overset{2}{D}\overset{1}{E}P$. *We need such formulas because our main aim is to investigate the commutators of the form* $[TrL^k(z),TrL^p(u)]$ *which can be represented by the formula:* $[TrA,TrB]=Tr[\overset{1}{A},\overset{2}{B}]$. *And it is easy to see that* $Tr\overset{1}{A}_1\overset{1}{A}_2...\overset{1}{A}_n\overset{2}{B}_1\overset{2}{B}_2...\overset{2}{B}_lP=Tr(A_1A_2...A_nB_1B_2...B_l)$, *but there is no simple formula for the expression like* $Tr\overset{1}{A}\overset{2}{B}\overset{1}{C}\overset{2}{D}\overset{1}{E}$.

On the basis of the formulas above we are able to obtain the following results on the commutativity of traces:

$$
\begin{aligned}
[TrL^k(z),TrL^l(u)] &= 0 \text{ for } k\le 3,\ l\le 3\forall z,u && (8)\\
[TrL^4(z),TrL^2(u)] &\neq 0 && (9)\\
Tr[L^k(z),L^l(u)] &= 0 \text{ for } k\le 3,\ l\le 2\forall z,u && (10)\\
Tr[L^4(z),L^2(u)] &\neq 0 && (11)\\
& && (12)
\end{aligned}
$$

(Let us mention that it can be seen from the proofs that $[TrL^k(z),TrL^l(u)]=0$ is more or less equivalent to the $Tr[L^k(z),L^{l-1}(u)]=0$.)

We pay special attention to the case of the one pole Gaudin model (the system of Mishchenko-Fomenko):

$$
L_{MF}(z)=\mathcal{K}+\frac{\Phi}{z} \qquad (13)
$$

Considerations of $\mathsf{symb}[TrL^k(z)]$ gives Poisson commutative subalgebra in $S(\mathfrak{gl}(n))$. This method is known as the "argument shift method" [11]. We analyze to what extent $TrL^k(z)$ can be used to obtain commutative subalgebras in $U(\mathfrak{gl}(n))$. We obtain that even in this simplest case there is no commutativity on the quantum level at least for $\mathfrak{gl}(n)$, $n\ge 6$:

$$
\begin{aligned}
[TrL^k_{MF}(z),TrL^2_{MF}(z)] &= 0\ \ \forall k && (14)\\
[TrL^k_{MF}(z),TrL^3_{MF}(z)] &= 0\ \ for\ \ k\le 5 && (15)\\
[TrL^6_{MF}(z),TrL^3_{MF}(z)] &\neq 0 && (16)
\end{aligned}
$$

We also prove that subalgebra generated by $TrL^k(z)$ contains the Cartan subalgebra, moreover it commutes with the Cartan subalgebra.

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2 Preliminaries

2.1 Leningrad's notation

Let us denote by P the transposition matrix in the tensor product, so it acts as follows

$$P(v_1 \otimes v_2) = v_2 \otimes v_1$$

and in terms of matrix elements it could be expressed as:

$$P = \sum_{i,j} e_{ij} \otimes e_{ji}.$$

Notation 1 *Leningrad's notation* $\overset{1}{T} = T \otimes Id, \overset{2}{T} = Id \otimes T$.

Always it is meant the following index notation for an element of $GL_n \otimes GL_n \otimes \mathcal{A}$ where $\mathcal{A}$ is an associative algebra

$$C = \sum_{ij,kl} C_{ij,kl} e_{ij} \otimes e_{kl}$$

where e_{ij} are generators of GL_n. Then $(\overset{1}{T})_{ij,kl} = T_{ij}\delta_{kl}$ and $(\overset{2}{T})_{ij,kl} = \delta_{ij}T_{kl}$.

Let us mention the following useful properties of such notation:

$$[\overset{1}{A}, \overset{2}{B}]_{ij,kl} = [A_{ij}, B_{kl}] \tag{17}$$

hence one matrix commutation relation between $\overset{1}{A}$ and $\overset{2}{B}$ encodes all the commutation relations between all matrix elements a_{ij} and b_{kl}. We will see later (see formula 26) that the commutation relations between basic elements of the Lie algebra $\mathfrak{gl}(n)$ are encoded in one simple relation, moreover this relation does not depend on n.

Let us mention the following useful properties of such notation:

$$P\overset{1}{A}\overset{2}{B} = \overset{2}{A}\overset{1}{B}P \tag{18}$$

$$P[\overset{1}{A}, P] = -[\overset{1}{A}, P]P = [P, \overset{1}{A}]P = [\overset{2}{A}, P]P = \overset{1}{A} - \overset{2}{A} \tag{19}$$

$$P[\overset{1}{A}, P] = [P, \overset{2}{A}]P \tag{20}$$

$$TrP\overset{1}{A}\overset{2}{B} = Tr\overset{1}{A}\overset{2}{B}P = TrP\overset{2}{A}\overset{1}{B} = Tr\overset{2}{A}\overset{1}{B}P = Tr(AB) \tag{21}$$

$$[TrA, TrB] = Tr[\overset{1}{A}, \overset{2}{B}] \qquad \text{(Main useful property I)} \tag{22}$$

$$Tr[A, B] = Tr[\overset{1}{A}, \overset{2}{B}]P \qquad \text{(Main useful property II)} \tag{23}$$

For any matrix $X_{ij,kl}$ in tensor square of some space V, the following is true:

$$(XP)_{ij,kl} = X_{il,kj} \tag{24}$$
$$(PX)_{ij,kl} = X_{kj,il} \tag{25}$$

2.2 Formulas without spectral parameter

Let us recall the basic facts from the r-matrix technique in the case of r-matrix without spectral parameter. We do this because our main aim will be to obtain analogous results in the case or L-operator of the Gaudin model with spectral parameter.

Introduce matrix Φ which is matrix with coefficients in $U(\mathfrak{gl}(n))$, defined by the following rule: we just put the element e_{ij} on ij-th place of the matrix. It's the simplest prototype of L-operators in integrable systems.

Lemma 1 *The commutation relations between elements $e_{ij} \in \mathfrak{gl}(n)$:*

$$[e_{ij}, e_{kl}] = e_{il}\delta_{jk} - e_{kj}\delta_{li}$$

can be encoded in the following way using Leningrad's notation:

$$[\overset{1}{\Phi}, \overset{2}{\Phi}] = [\overset{1}{\Phi}, P] = [P, \overset{2}{\Phi}] = \frac{1}{2}[\overset{1}{\Phi} - \overset{2}{\Phi}, P] \tag{26}$$

This commutation relation is of the so-called "r-matrix" type without spectral parameter. Matrix P is the simplest example of the classical r-matrix.

Corollary 1 *The formula above can be rewritten as*

$$[\overset{1}{\Phi} - \frac{1}{2}P, \overset{2}{\Phi} - \frac{1}{2}P] = 0 \tag{27}$$

this can be used to obtain formulas like $[(\overset{1}{\Phi} - \frac{1}{2}P)^l, (\overset{2}{\Phi} - \frac{1}{2}P)^n] = 0$, but we will not use this here.

For any matrix Φ satisfying the formula in lemma 1 the following is true:

Lemma 2 *By the Leibnitz rule one immediately obtains:*

$$[\overset{1}{\Phi}{}^n, \overset{2}{\Phi}] = [\overset{1}{\Phi}{}^n, P] = [P, \overset{2}{\Phi}{}^n] \tag{28}$$

Lemma 3 *Using the formula above one obtains:*

$$[\overset{1}{\Phi}{}^r, \overset{2}{\Phi}{}^s] = \sum_{a=1}^{min(r,s)} P\overset{1}{\Phi}{}^{a-1}\overset{2}{\Phi}{}^{r+s-a} - P\overset{1}{\Phi}{}^{r+s-a}\overset{2}{\Phi}{}^{a-1} \tag{29}$$

As a demonstration of the "r-matrix" technique we prove the following statement, which is due to Gelfand [21].

Lemma 4 *The elements $Tr\Phi^r \in U(\mathfrak{gl}(n))$ lie in the center of the $U(\mathfrak{gl}(n))$, i.e. they are Casimirs of $U(\mathfrak{gl}(n))$.*
Proof

$$[Tr\Phi^r, \Phi_{i,j}] = \sum_k ([\overset{1}{\Phi}{}^r, \overset{2}{\Phi}])_{kk,ij} = \sum_k ([\overset{1}{\Phi}{}^r, P])_{kk,ij} = \sum_k (\overset{1}{\Phi}{}^r P - P\overset{1}{\Phi}{}^r)_{kk,ij} =$$
$$= \sum_k (\overset{1}{\Phi}{}^r)_{kj,ik} - (\overset{1}{\Phi}{}^r)_{ik,kj} = \sum_k (\Phi^r_{kj}\delta_{ik} - \Phi^r_{ik}\delta_{kj}) = (\Phi^r_{i,j} - \Phi^r_{i,j}) = 0 \qquad (30)$$

□

Remark 2 In fact $Tr\Phi^r$, $r = 1, ..., n$ generate the center of $U(\mathfrak{gl}(n))$ (see [21]).

Lemma 5 *One can also prove the following (see for example [18]):*

$$[\overset{1}{\Phi}{}^{r+1}, \overset{2}{\Phi}{}^s] - [\overset{1}{\Phi}{}^r, \overset{2}{\Phi}{}^{s+1}] = P\overset{1}{\Phi}{}^r\overset{2}{\Phi}{}^s - P\overset{1}{\Phi}{}^s\overset{2}{\Phi}{}^r \qquad (31)$$

2.3 Gaudin model and its Lax operator with spectral parameter

Let us recall the Gaudin model. Consider the direct sum $\mathfrak{gl}(n) \oplus ... \oplus \mathfrak{gl}(n)$ and denote by Φ_i the matrix $n \times n$ with values in this direct sum defined as follows: $(\Phi_i)_{kl}$ is the e_{kl} generator of the i-th copy of $\mathfrak{gl}(n)$ in the direct sum $\mathfrak{gl}(n) \oplus ... \oplus \mathfrak{gl}(n)$. Let us introduce the Lax operator for the Gaudin model:

$$L(z) = \mathcal{K} + \sum_{i=1...N} \frac{\Phi_i}{z - a_i} \qquad (32)$$

where $a_i \in \mathbb{C}$, $\mathcal{K}$ is an arbitrary constant matrix (in original physical applications the case of $sl(2)$ was important, in this case variables Φ corresponds to spins and $\mathcal{K}$ corresponds to magnetic field. Nowadays the range of physical applications of integrable spin chains is quite reach and includes not only $sl(2)$ case, see for example [22, 23]).
Let us consider

$$TrL^2(z) = \sum_i \frac{Tr\Phi_i^2}{(z - a_i)^2} + \sum_k \frac{1}{z - a_k}(Tr\mathcal{K}\Phi_k + \sum_{j\neq k} \frac{2Tr\Phi_k\Phi_j}{(a_k - a_j)}) \qquad (33)$$

$$H_k = Tr\mathcal{K}\Phi_k + \sum_{j\neq k} \frac{2Tr\Phi_k\Phi_j}{(a_k - a_j)} \in U(\mathfrak{gl}(n)) \otimes ... \otimes U(\mathfrak{gl}(n)) \qquad (34)$$

- are called quadratic Gaudin Hamiltonians

It's known that $[TrL^2(u), TrL^2(z)] = 0$ and so that $[H_k, H_j] = 0$. The Gaudin model on quantum level consists of taking some representation $V_1 \otimes ... \otimes V_N$ of $\mathfrak{gl}(n) \oplus ... \oplus \mathfrak{gl}(n)$ and asking for the spectrum of such operators, their matrix elements etc.

On the classical level one uses the symbol map $\mathsf{symb} : U(\mathfrak{gl}(n)) \to S(\mathfrak{gl}(n))$. The images of H_i can be interpreted as functions on $\mathfrak{gl}(n)^* \oplus ... \oplus \mathfrak{gl}^*(n)$. One restricts them to the product of coadjoint orbits $\mathcal{O}_1 \times ... \times \mathcal{O}_N$ which is by definition the phase space of classical Gaudin model. The symplectic structure on this space is Kirillov's symplectic form. For the hamiltonian one can take any of functions $\mathsf{symb}(H_i)$. The classical Gaudin model is known to be completely integrable: one can take coefficients at $(z-a_i)^{-m}$ of $\mathsf{symb}(TrL^k(z))$ as hamiltonians which Poisson commute and at least for the general choice of orbits (though it seems that the question of considering different non-general orbits has not been analyzed carefully) this gives the family of independent hamiltonians. Their number equals to the half the dimension of the phase space which means the complete integrability in the Liouville sense.

More mathematically speaking one should mention that Gaudin hamiltonians generate the maximal Poisson-commutative subalgebra in $S(\mathfrak{gl}(n)) \otimes ... \otimes S(\mathfrak{gl}(n))$, though it's difficult to give the reference for this result.

The Lax-pair representation for the Gaudin model on the both quantum and classical level can be found in [24]. The Gaudin Lax operator was generalized to the trigonometric [24] and elliptic [25] dependence on the spectral parameter. It was also generalized to include higher poles in z see [26, 27, 28].

3 Commutativity and noncommutativity of traces

3.1 Commutation relations with spectral parameter

As it was mentioned before it's known that Gaudin model is completely integrable on the classical and quantum levels, moreover the commuting hamiltonians on the classical level can be given as coefficients at $(z-a_i)^{-m}$ of $\mathsf{symb}(TrL(z)^k)$, but no explicit formula for the higher than quadratic quantum integrals is known. Our aim is to check whether the $TrL(z)^k$ commute on the quantum level. Before doing this it is necessary to obtain the commutation relations between powers of $L(z)$, generalizing such relations in the case without spectral parameter see formulas 28,29.

Lemma 6 *Using formula 26 one immediately obtains:*

$$\begin{aligned}
[\overset{1}{L}(z), \overset{2}{L}(u)] = [\overset{1}{L}(u), \overset{2}{L}(z)] = [\frac{P}{z-u}, \overset{1}{L}(z) + \overset{2}{L}(u)] = \\
= [\frac{P}{z-u}, \overset{1}{L}(z) - \overset{1}{L}(u)] = [\frac{P}{z-u}, -\overset{2}{L}(z) + \overset{2}{L}(u)]
\end{aligned} \tag{35}$$

This commutation relation is of the so-called linear r-matrix type. The matrix $\frac{P}{z-u}$ is the simplest rational r-matrix. In most of known integrable systems the Lax operator satisfies an analogous relation (with the other r-matrices and possibly with the "linear" relation changed to the "quadratic" one). These relations provide a simple construction on the classical and sometimes on the quantum level.

Corollary 2 *The formula above can be rewritten as*

$$[\overset{1}{L}(z) - \frac{P}{z-u}, \overset{2}{L}(u) + \frac{P}{z-u}] = 0. \tag{36}$$

Lemma 7 *The r-matrix $r(z,u) = \frac{P}{z-u}$ satisfies classical Yang-Baxter equation:*

$$[r_{12}(u_1,u_2), r_{13}(u_1,u_3) + r_{23}(u_2,u_3)] - [r_{13}(u_1,u_3), r_{32}(u_3,u_2)] = 0 \in V \otimes V \otimes V \tag{37}$$

where we use the standard notation r_{ij} which means linear operator in $V \otimes V \otimes V$ which act as operator r, but the action is on i-th and j-th components of the tensor product $V \otimes V \otimes V$, for example $P_{13}(a \otimes b \otimes c) = (c \otimes b \otimes a)$.

Remark 3 There is some confusion in notation: one says "classical r-matrix", "classical Yang-Baxter equation" though we deal with the quantum problem (commutators instead of Poisson brackets). This is due to the historical reasons: in the case of quadratic commutation relations (which seems to be appeared before linear relations) quantum R-matrix corresponds to the quantum case and classical to the classical case. In the case of linear commutation relations classical Yang-Baxter equation and classical r-matrix corresponds to both classical and quantum cases.

To go further we need for several technical lemmas:

Lemma 8 *For the Gaudin Lax it's true that:*

$$[\overset{1}{L}(u), \overset{2}{L}(u)] = [P, \frac{\partial}{\partial u}\overset{1}{L}(u)] \tag{38}$$

Lemma 9 *One has the following expressions for commutators:*

$$\begin{aligned}
[\overset{1}{L}(z), \overset{2}{L}(u)] &= (\overset{1}{L}(u) - \overset{2}{L}(u))\frac{P}{z-u} + (\overset{2}{L}(z) - \overset{1}{L}(z))\frac{P}{z-u} && (39)\\
[\overset{1}{L}(z), \overset{2}{L}{}^2(u)] &= (\overset{1}{L}{}^2(u) - \overset{2}{L}{}^2(u))\frac{P}{z-u} + \\
&\quad \overset{2}{L}(u)(\overset{2}{L}(z) - \overset{1}{L}(z))\frac{P}{z-u} + (\overset{2}{L}(z) - \overset{1}{L}(z))\overset{1}{L}(u)\frac{P}{z-u} && (40)\\
[\overset{1}{L}(z), \overset{2}{L}{}^n(u)] &= (\overset{1}{L}{}^n(u) - \overset{2}{L}{}^n(u))\frac{P}{z-u} + \\
&\quad \sum_{i=0}^{n-1} \overset{2}{L}{}^i(u)(\overset{2}{L}(z) - \overset{1}{L}(z))\overset{1}{L}{}^{n-1-i}(u)\frac{P}{z-u} && (41)
\end{aligned}$$

Lemma 10 *The expressions of the previous lemma can be rewritten in the ordered form*[5]*:*

$$
\begin{aligned}
[\overset{1}{L}(z), \overset{2}{L}(u)] &= (\overset{1}{L}(u) - \overset{2}{L}(u))\frac{P}{z-u} + (\overset{2}{L}(z) - \overset{1}{L}(z))\frac{P}{z-u} && (42)\\
[\overset{1}{L}{}^2(z), \overset{2}{L}(u)] &= (\overset{2}{L}{}^2(z) - \overset{1}{L}{}^2(z))\frac{P}{z-u} + \\
&\quad \overset{1}{L}(z)(\overset{1}{L}(u) - \overset{2}{L}(u))\frac{P}{z-u} + (\overset{1}{L}(u) - \overset{2}{L}(u))\overset{2}{L}(z)\frac{P}{z-u} && (43)\\
[\overset{1}{L}{}^n(z), \overset{2}{L}(u)] &= (\overset{2}{L}{}^n(z) - \overset{1}{L}{}^n(z))\frac{P}{z-u} + \\
&\quad \sum_{i=0}^{n-1} \overset{1}{L}{}^i(z)(\overset{1}{L}(u) - \overset{2}{L}(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u} && (44)
\end{aligned}
$$

Receipt to obtain the formulas for $[L^n(z), L^m(u)]$**.** Let us formulate the main necessary observation which provides such formulas. It is easy to obtain the commutation relation between $\overset{1}{L}{}^n(z), \overset{2}{L}(u)$ in the form presented above: i.e. as the sum terms of the form $\overset{1}{A}_1 ... \overset{1}{A}_p \overset{2}{B}_1 ... \overset{2}{B}_q P$, but it is difficult to obtain the same answer as the sum of terms of the form $\overset{2}{B}_1 ... \overset{2}{B}_q \overset{1}{A}_1 ... \overset{1}{A}_p P$. *As soon as one is able to find such formula then the formula for the commutator of* $\overset{1}{L}{}^n(z), \overset{2}{L}{}^m(u)$ *follows immediately.*

The reason is quite simple: assume that $[\overset{1}{A}, \overset{2}{B}] = \overset{1}{C}\overset{2}{D}P$ then

$$[\overset{1}{A}{}^n, \overset{2}{B}] = \sum_{i=0}^{n} \overset{1}{A}{}^i \overset{1}{C}\overset{2}{D} P \overset{1}{A}{}^{n-1-i} = \sum_{i=0}^{n} \overset{1}{A}{}^i \overset{1}{C}\overset{2}{D}\overset{2}{A}{}^{n-1-i}.P$$

One can see that the trick does not work if we have the formula like: $[\overset{1}{A}, \overset{2}{B}] = \overset{2}{X}\overset{1}{Y}P$. In this case

$$[\overset{1}{A}{}^n, \overset{2}{B}] = \sum_{i=0}^{n} \overset{1}{A}{}^i \overset{2}{X}\overset{1}{Y}\overset{2}{A}{}^{n-1-i} P$$

and one is unable to calculate the trace of such expression in the simple form.

So in order to calculate the $[L^n(z), L^2(u)]$ (as explained above) we need to find the formula for the $[L^n(z), L^2(u)]$ in such form that it is the sum of the type $\overset{1}{A}\overset{2}{B}P$. It's done in the following lemma.

[5] Our aim is to obtain the formula for $[L^n(z), L^m(u)]$ in the form which is the sum of terms like

$$\overset{1}{A}_1 \overset{1}{A}_2 ... \overset{1}{A}_n \overset{2}{B}_1 \overset{2}{B}_2 ... \overset{2}{B}_l P^{0 \ or \ 1}$$

and there are no terms of the disordered type: $Tr\overset{1}{A}\overset{2}{B}\overset{1}{C}\overset{2}{D}\overset{1}{E}P$. We need such formulas because they are well adapted to calculation of $[TrL^k(z), TrL^p(u)]$, indeed, $[TrA, TrB] = Tr[\overset{1}{A}, \overset{2}{B}]$ and it is easy to see that

$$Tr\overset{1}{A}_1 \overset{1}{A}_2 ... \overset{1}{A}_n \overset{2}{B}_1 \overset{2}{B}_2 ... \overset{2}{B}_l P = Tr(A_1 A_2 ... A_n B_1 B_2 ... B_l)$$

but there is no simple formula for the expression like $Tr\overset{1}{A}\overset{2}{B}\overset{1}{C}\overset{2}{D}\overset{1}{E}$.

Lemma 11

$$
\begin{aligned}
[\overset{1}{L}(z),\overset{2}{L}{}^2(u)] &= (\overset{1}{L}{}^2(u)-\overset{2}{L}{}^2(u))\frac{P}{z-u} \\
&+ (\overset{1}{L}(u)+\overset{2}{L}(u))\overset{2}{L}(z)\frac{P}{z-u}-\overset{1}{L}(z)(\overset{1}{L}(u)+\overset{2}{L}(u))\frac{P}{z-u}
\end{aligned} \tag{45}
$$

Lemma 12 *From the lemma above one obtains the following formula:*

$$
\begin{aligned}
[\overset{1}{L}{}^n(z),\overset{2}{L}{}^2(u)] &= (\overset{1}{L}(u)+\overset{2}{L}(u))\overset{2}{L}{}^n(z)\frac{P}{z-u}-\overset{1}{L}{}^n(z)(\overset{1}{L}(u)+\overset{2}{L}(u))\frac{P}{z-u}+ \\
&+ \sum_{i=0}^{n-1}\overset{1}{L}{}^i(z)(\overset{1}{L}{}^2(u)-\overset{2}{L}{}^2(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u}
\end{aligned} \tag{46}
$$

Our next aim is to calculate $[L^n(z),L^3(u)]$. As it was explained in our receipt we need to find the formula for the $[L(z),L^3(u)]$ in such form that it is the sum of the type $\overset{1}{A}\overset{2}{B}P$. It's done in the following lemma.

Lemma 13

$$
\begin{aligned}
[\overset{1}{L}(z),\overset{2}{L}{}^3(u)] &= (\overset{1}{L}{}^3(u)-\overset{2}{L}{}^3(u))\frac{P}{z-u} \\
&+ (\overset{1}{L}{}^2(u)+\overset{1}{L}(u)\overset{2}{L}(u)+\overset{2}{L}{}^2(u))\overset{2}{L}(z)\frac{P}{z-u} \\
&- \overset{1}{L}(z)(\overset{1}{L}{}^2(u)+\overset{1}{L}(u)\overset{2}{L}(u)+\overset{2}{L}{}^2(u))\frac{P}{z-u} \\
&+ [\frac{\partial}{\partial u}\overset{1}{L}(u)+\frac{\overset{1}{L}(u)}{z-u},\overset{1}{L}(z)]\frac{1}{z-u}
\end{aligned} \tag{47}
$$

Going ahead let us note that as a corollary of the lemma above we see that $Tr[L^2(u),L(z)]=0$.

Lemma 14 *From the lemma above one also sees that:*

$$
\begin{aligned}
[\overset{1}{L}{}^n(z),\overset{2}{L}{}^3(u)] &= \sum_{i=0}^{n-1}\overset{1}{L}{}^i(z)(\overset{1}{L}{}^3(u)-\overset{2}{L}{}^3(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u} \\
&+ (\overset{1}{L}{}^2(u)+\overset{1}{L}(u)\overset{2}{L}(u)+\overset{2}{L}{}^2(u))\overset{2}{L}{}^n(z)\frac{P}{z-u} \\
&- \overset{1}{L}{}^n(z)(\overset{1}{L}{}^2(u)+\overset{1}{L}(u)\overset{2}{L}(u)+\overset{2}{L}{}^2(u))\frac{P}{z-u} \\
&+ [\frac{\partial}{\partial u}\overset{1}{L}(u)+\frac{\overset{1}{L}(u)}{z-u},\overset{1}{L}{}^n(z)]\frac{1}{z-u}
\end{aligned} \tag{48}
$$

Remark 4 *In general one can guess the following sort of the formula:*

$$\begin{aligned}
[\overset{1}{L}{}^n(z), \overset{2}{L}{}^m(u)] &= \sum_{i=0}^{n-1} \overset{1}{L}{}^i(z)(\overset{1}{L}{}^m(u) - \overset{2}{L}{}^m(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u} \\
&+ (\sum_{i=0}^{m-1} \overset{1}{L}{}^i(u)\overset{2}{L}{}^{m-1-i}(u))\overset{2}{L}{}^n(z)\frac{P}{z-u} \\
&- \overset{1}{L}{}^n(z)(\sum_{i=0}^{m-1} \overset{1}{L}{}^i(u)\overset{2}{L}{}^{m-1-i}(u))\frac{P}{z-u} \\
&+ \text{“unwanted terms of lower degree”}
\end{aligned} \tag{49}$$

But even in the case of m=4 it seems to be quite difficult to obtain such formula.

3.2 Commutativity of traces $[TrL^n(z), TrL^m(u)] = 0$, $n, m \leq 3$

We will prove that $[TrL^n(z), TrL^m(u)] = 0$, for $n \leq 3, m \leq 3$, and speculate about the general case. The case $n = 2 = m$ is easy and quite well-known. In the next section we will prove that $[TrL^4(z), TrL^2(u)] \neq 0$.

Lemma 15 *The equality* $[TrL^n(z), TrL^2(u)] = 0$ *is equivalent to the equality* $Tr[L^n(z), L(u)] = 0$*, moreover* $[TrL^n(z), TrL^2(u)] = 2Tr[L(u), L^n(z)]\frac{1}{z-u}$*.*
Proof

$$\begin{aligned}
& [TrL^n(z), TrL^2(u)] = Tr[\overset{1}{L}{}^n(z), \overset{2}{L}{}^2(u)] = \text{ / by lemma 12 /} \\
= \; & Tr\Big((\overset{1}{L}(u) + \overset{2}{L}(u))\overset{2}{L}{}^n(z)\frac{P}{z-u} - \overset{1}{L}{}^n(z)(\overset{1}{L}(u) + \overset{2}{L}(u))\frac{P}{z-u} \\
+ \; & \sum_{i=0}^{n-1} \overset{1}{L}{}^i(z)(\overset{1}{L}{}^2(u) - \overset{2}{L}{}^2(u))\overset{2}{L}{}^{n-1-i}(z)\Big) = \text{by the formula 21} \\
= \; & 2Tr(L(u)L^n(z))\frac{1}{z-u} - 2Tr(L^n(z)L(u))\frac{1}{z-u} + \\
+ \; & \sum_{i=0}^{n-1} TrL^i(z)L^2(u)L^{n-1-i}(z) - TrL^i(z)L^2(u)L^{n-1-i}(z) \\
= \; & 2Tr[L(u), L^n(z)]\frac{1}{z-u}
\end{aligned} \tag{50}$$

□

Lemma 16 *The following equality holds:*

$$\begin{aligned}
Tr[L^n(z), L(u)] &= \frac{1}{z-u}\sum_{i=1}^{n-1}\Big(TrL^i(z)L(u)TrL^{n-1-i}(z) - TrL^{n-1-i}(z)TrL(u)L^i(z)\Big) \\
&= \frac{1}{z-u}\Big(\sum_{i=1}^{n-1}[TrL^i(z)L(u), TrL^{n-1-i}(z)] - \sum_{i=0}^{n-2} TrL^i(z)Tr[L(u), L^{n-1-i}(z)]\Big)
\end{aligned} \tag{51}$$

Proof This is a straightforward application of formula 44 and formulas $Tr\overset{1}{A}\overset{2}{B} = TrATrB$ and $Tr[A,B] = Tr[\overset{1}{A}\overset{2}{B}]P$:

$$\begin{aligned}
& Tr[L^n(z), L(u)] = Tr[\overset{1}{L}{}^n(z), \overset{2}{L}(u)]P = \text{ by the formula 44} \\
= \; & Tr\Big((\overset{2}{L}{}^n(z) - \overset{1}{L}{}^n(z)) + \sum_{i=0}^{n-1} \overset{1}{L}{}^i(z)(\overset{1}{L}(u) - \overset{2}{L}(u))\overset{2}{L}{}^{n-1-i}(z)\Big)\frac{1}{z-u} \\
= \; & \Big((TrId)(TrL^n(z) - TrL^n(z)) \\
+ \; & \sum_{i=0}^{n-1} TrL^i(z)L(u)TrL^{n-1-i}(z) - TrL^i(z)TrL(u)L^{n-1-i}(z)\Big)\frac{1}{z-u} \\
= \; & \Big(\sum_{i=0}^{n-1} TrL^i(z)L(u)TrL^{n-1-i}(z) - \sum_{i=0}^{n-1} TrL^{n-1-i}(z)TrL(u)L^i(z)\Big)\frac{1}{z-u} \\
& \text{when i=0 then } TrL(u)TrL^{n-1}(z) - TrL^{n-1}(z)TrL(u) = 0 \\
& \text{due to } TrL(z) \text{ lies in the center of } \mathfrak{gl}(n) \oplus ... \oplus \mathfrak{gl}(n) \\
= \; & \Big(\sum_{i=1}^{n-1} TrL^i(z)L(u)TrL^{n-1-i}(z) - TrL^{n-1-i}(z)TrL(u)L^i(z)\Big)\frac{1}{z-u} \\
= \; & \frac{1}{z-u}\Big(\sum_{i=1}^{n-1}[TrL^i(z)L(u), TrL^{n-1-i}(z)] - \sum_{i=1}^{n-1} TrL^{n-1-i}(z)Tr[L(u), L^i(z)]\Big) \\
= \; & \frac{1}{z-u}\Big(\sum_{i=1}^{n-1}[TrL^i(z)L(u), TrL^{n-1-i}(z)] - \sum_{i=0}^{n-2} TrL^i(z)Tr[L(u), L^{n-1-i}(z)]\Big) \quad (52)
\end{aligned}$$

□

Corollary 3

$$Tr[L^n(z), L(u)] = 0 \; \textit{for n=1,2,3} \tag{53}$$

Proof
Let $n = 1$ - then by the lemma above we immediately obtain zero, because the summation index i is out of range in the sum $\sum_{i=1}^{n-1}$.
Let $n = 2$, so by the lemma above

$$\begin{aligned}
& (z-u)Tr[L^2(z), L(u)] = \sum_{i=1}^{1} TrL^i(z)L(u)TrL^{n-1-i}(z) - TrL^{n-1-i}(z)TrL(u)L^i(z) \\
= \; & (TrId)Tr[L(z), L(u)] = 0 \quad \text{in virtue of the corollary in the case } n = 1
\end{aligned}$$

Let $n = 3$, so by the lemma above:

$$\begin{aligned}
(z-u)Tr[L^3(z), L(u)] &= \sum_{i=1}^{2} TrL^i(z)L(u)TrL^{n-1-i}(z) - TrL^{n-1-i}(z)TrL(u)L^i(z) \\
&= \big(TrL(z)L(u)TrL(z) - TrL(z)TrL(u)L(z)\big) \\
&+ \big(TrL^2(z)L(u)(Tr(Id)) - (Tr(Id))TrL(u)L^i(z)\big) = 0 \\
&\quad \text{in virtue of the collorary in the case } n = 1,\ n = 2 \\
&\quad \text{and the commutativity of } TrL(z) \text{ with everything.}
\end{aligned}$$

□

Corollary 4 *One can simplify (reducing the range of summation) the expression in lemma 16 in the following way:*

$$\begin{aligned}
Tr[L^n(z), L(u)] &= \frac{1}{z-u}\Big(\sum_{i=1}^{n-3}[TrL^i(z)L(u), TrL^{n-1-i}(z)] \\
&- \sum_{i=0}^{n-5} TrL^i(z)Tr[L(u), L^{n-1-i}(z)]\Big)
\end{aligned} \tag{54}$$

Corollary 5

$$Tr[L^4(z), L(u)] = \frac{1}{z-u}[TrL(z)L(u), TrL^2(z)] \tag{55}$$

Corollary 6

$$[TrL^n(z), TrL^2(u)] = 0 \textit{ for } n = 1, 2, 3 \tag{56}$$

Proof This follows from the corollary (3) and lemma 15. □

Lemma 17 *The equality* $[TrL^n(z), TrL^3(u)] = 0$ *follows from the equalities* $Tr[L^n(z), L^2(u)] = 0$ *and* $Tr[L^n(z), L(u)] = 0$*, moreover*

$$\begin{aligned}
&[TrL^n(z), TrL^3(u)] = \\
&= \Big(3Tr[L^2(u), L^n(z)]\frac{1}{z-u} + (TrId)Tr[\frac{\partial}{\partial u}L(u) + \frac{L(u)}{z-u}, L^n(z)]\frac{1}{z-u}\Big).
\end{aligned} \tag{57}$$

Proof

$$
\begin{aligned}
& [TrL^n(z), TrL^3(u)] = Tr[\overset{1}{L}{}^n(z), \overset{2}{L}{}^3(u)] = /\text{ by lemma 14 }/ \\
= \; & Tr\Big(\sum_{i=0}^{n-1} \overset{1}{L}{}^i(z)(\overset{1}{L}{}^3(u) - \overset{2}{L}{}^3(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u} \\
+ \; & (\overset{1}{L}{}^2(u) + \overset{1}{L}(u)\overset{2}{L}(u) + \overset{2}{L}{}^2(u))\overset{2}{L}{}^n(z)\frac{P}{z-u} \\
- \; & \overset{1}{L}{}^n(z)(\overset{1}{L}{}^2(u) + \overset{1}{L}(u)\overset{2}{L}(u) + \overset{2}{L}{}^2(u))\frac{P}{z-u} \\
+ \; & [\frac{\partial}{\partial u}\overset{1}{L}(u) + \frac{\overset{1}{L}(u)}{z-u}, \overset{1}{L}{}^n(z)]\frac{1}{z-u}\Big) \\
= \; & \Big(3(TrL^2(u)L^n(z)\frac{1}{z-u} - TrL^n(z)L^2(u)\frac{1}{z-u}) + \\
+ \; & (TrId)Tr[\frac{\partial}{\partial u}L(u) + \frac{L(u)}{z-u}, L^n(z)]\frac{1}{z-u}\Big) \\
= \; & \Big(3Tr[L^2(u), L^n(z)]\frac{1}{z-u} + (TrId)Tr[\frac{\partial}{\partial u}L(u) + \frac{L(u)}{z-u}, L^n(z)]\frac{1}{z-u}\Big)
\end{aligned}
$$

□

Corollary 7 *The following is true:*

$$Tr[L^2(z), L^2(u)] = 0 \tag{58}$$

Proof This follows from corollary 6 that $[TrL^2(z), TrL^3(u)] = 0$ and from corollary 3 that $Tr[L(u), L^2(z)] = 0$, so according to the lemma above formula 57 applied to the case $n = 2$ gives that:

$$0 = \frac{3}{z-u}[TrL^2(u), TrL^2(z)] + 0$$

□

Lemma 18 *The following equality holds:*

$$
\begin{gathered}
(z-u)Tr[L^n(z), L^2(u)] = \\
= Tr[L(u), L^n(z)] + \textstyle\sum_{i=0}^{n-3}[TrL^i(z)L^2(u), TrL^{n-1-i}(z)] - TrL^i(z)Tr[L^2(u), L^{n-1-i}(z)]
\end{gathered}
$$

Proof

$$
\begin{aligned}
& Tr[L^n(z), L^2(u)] = Tr[\overset{1}{L}{}^n(z), \overset{2}{L}{}^2(u)]P = / \text{ by lemma 12 } / \\
= \;& Tr\Big((\overset{1}{L}(u) + \overset{2}{L}(u))\overset{2}{L}{}^n(z)\frac{1}{z-u} - \overset{1}{L}{}^n(z)(\overset{1}{L}(u) + \overset{2}{L}(u))\frac{1}{z-u} \\
+ \;& \sum_{i=0}^{n-1} \overset{1}{L}{}^i(z)(\overset{1}{L}{}^2(u) - \overset{2}{L}{}^2(u))\overset{2}{L}{}^{n-1-i}(z)\frac{1}{z-u}\Big) \\
= \;& \frac{1}{z-u}\Big([TrL(u), TrL^n(z)] + Tr[L(u), L^n(z)] \\
+ \;& \sum_{i=0}^{n-1} TrL^i(z)L^2(u)TrL^{n-1-i}(z) - TrL^i(z)TrL^2(u)L^{n-1-i}(z)\Big) \\
= \;& \frac{1}{z-u}\Big(Tr[L(u), L^n(z)] + \sum_{i=0}^{n-1} TrL^i(z)L^2(u)TrL^{n-1-i}(z) \\
- \;& TrL^{n-1-i}(z)TrL^2(u)L^i(z)\Big) \\
= \;& \frac{1}{z-u}\Big(Tr[L(u), L^n(z)] + \sum_{i=0}^{n-1}[TrL^i(z)L^2(u), TrL^{n-1-i}(z)] \\
- \;& TrL^{n-1-i}(z)Tr[L^2(u), L^i(z)]\Big) \text{ by corollaries 3, 7} \\
= \;& \frac{1}{z-u}\Big(Tr[L(u), L^n(z)] + \sum_{i=0}^{n-3}[TrL^i(z)L^2(u), TrL^{n-1-i}(z)] \\
- \;& TrL^i(z)Tr[L^2(u), L^{n-1-i}(z)]\Big) \qquad (59)
\end{aligned}
$$

□

Corollary 8 *The following is true:*

$$Tr[L^3(z), L^2(u)] = 0 \qquad (60)$$

Proof According to the lemma above we obtain:

$$
\begin{aligned}
& (z-u)Tr[L^3(z), L^2(u)] \\
= \;& Tr[L(u), L^3(z)] + \sum_{i=0}^{0}[TrL^i(z)L^2(u), TrL^{2-i}(z)] - TrL^i(z)Tr[L^2(u), L^{2-i}(z)] \\
= \;& Tr[L(u), L^3(z)] + [TrL^2(u), TrL^2(z)] - TrIdTr[L^2(u), L^2(z)] = 0 \qquad (61)
\end{aligned}
$$

in virtue of corollaries 3, 6, 7. □

Corollary 9 *The following is true:*

$$[TrL^3(z), TrL^3(u)] = 0 \qquad (62)$$

Proof this is directly implied by lemma 17 because the conditions requested in this lemma are true by corollaries 3, 8 □

3.3 NONcommutativity of traces L^4, L^2

According to lemma 15 and corollary 5

$$[TrL^4(z), TrL^2(u)] = \frac{2}{z-u} Tr[L(u), L^4(z)] = \frac{-2}{(z-u)^2}[TrL(z)L(u), TrL^2(z)] \tag{63}$$

Let us consider for simplicity the case $\mathcal{K} = 0$.

$$TrL^2(z) = \sum_i \frac{Tr\Phi_i^2}{(z-a_i)^2} + \sum_k \frac{1}{z-a_k} \sum_{j\neq k} \frac{2Tr\Phi_k\Phi_j}{(a_k-a_j)} \tag{64}$$

$$H_k = \sum_{j\neq k} \frac{2Tr\Phi_k\Phi_j}{(a_k-a_j)} \tag{65}$$

It follows from $[TrL^2(u), TrL^2(z)] = 0$ that $[H_k, H_j] = 0$.

$$TrL(z)L(u) = \sum_i \frac{Tr\Phi_i^2}{(z-a_i)(u-a_i)} + \sum_m \frac{1}{u-a_m} \sum_{n\neq m} \frac{Tr\Phi_m\Phi_n}{z-a_n} \tag{66}$$

The commutator $[TrL(z)L(u), TrL^2(z)]$ vanishes if

$$0 = [\sum_m \frac{1}{z-a_m} \sum_{n\neq m} \frac{Tr\Phi_m\Phi_n}{u-a_n}, \sum_k \frac{1}{u-a_k} \sum_{j\neq k} \frac{2Tr\Phi_k\Phi_j}{(a_k-a_j)}] \tag{67}$$

This expression is a rational function on variables z, u. It is zero iff its residues on z are zeros. For example, its residue at $z = a_m$ is a rational function on u with double poles at $u = a_i$, let us consider its coefficient at $\frac{1}{(u-a_n)^2}$

$$[Tr\Phi_m\Phi_n, \sum_{j\neq n} \frac{2Tr\Phi_n\Phi_j}{(a_n-a_j)}] \tag{68}$$

This could be zero for any a_k only if the expression below should be zero for all $m, n, j \neq m$

$$\begin{aligned}
&[Tr\Phi_m\Phi_n, Tr\Phi_n\Phi_j] = Tr[\overset{1}{\Phi}_m\overset{1}{\Phi}_n, \overset{2}{\Phi}_n\overset{2}{\Phi}_j] = Tr\overset{1}{\Phi}_m[\overset{1}{\Phi}_n, \overset{2}{\Phi}_n]\overset{2}{\Phi}_j = Tr\overset{1}{\Phi}_m[\overset{1}{\Phi}_n, P]\overset{2}{\Phi}_j \\
&= Tr\overset{1}{\Phi}_m\overset{1}{\Phi}_n\overset{1}{\Phi}_jP - TrP\overset{2}{\Phi}_m\overset{1}{\Phi}_n\overset{2}{\Phi}_j = Tr\overset{1}{\Phi}_m\overset{1}{\Phi}_n\overset{1}{\Phi}_jP - TrP\overset{2}{\Phi}_m\overset{2}{\Phi}_j\overset{1}{\Phi}_n \\
&= Tr\Phi_m\Phi_n\Phi_j - Tr\Phi_m\Phi_j\Phi_n \neq \text{ZERO !!!}
\end{aligned} \tag{69}$$

Let us note that there is no such a problem for the case of one- or two-poles Gaudin system.

So we obtain that

$$0 \neq [TrL^4(z), TrL^2(u)] = \frac{2}{z-u} Tr[L(u), L^4(z)] = \frac{-2}{(z-u)^2}[TrL(z)L(u), TrL^2(z)] \tag{70}$$

□

3.4 One spin Gaudin model (argument shift method)

The one spin Gaudin's model Lax operator is the following one:

$$L_{MF}(z) = \mathcal{K} + \frac{\Phi}{z} \tag{71}$$

It's well-known that considering the coefficients at z^{-1} of $\mathsf{symb}(TrL^k_{MF}(z)) \in S(\mathfrak{gl}(n))$ one obtains the maximal Poisson-commutative subalgebra in $S(\mathfrak{gl}(n))$ (for general $\mathcal{K}$) [11, 15]. The restriction of such subalgebra to a generic coadjoint orbit gives a classical integrable system.

The aim of this section is to discuss the lifting of this maximal commutative subalgebra from $S(\mathfrak{gl}(n))$ to $U(\mathfrak{gl}(n))$, by considering $TrL^k_{MF}(z) \in U(\mathfrak{gl}(n))$. In other words we discuss the quantum integrability of the one spin Gaudin model. We prove commutativity in the following restricted case:

$$[TrL^k_{MF}(z)), TrL^l_{MF}(u))] = 0 \qquad \text{for} \qquad k, l \leq 4$$

this is more optimistic than the general Gaudin model, however we also prove that

$$[TrL^3_{MF}(z)), TrL^6_{MF}(u))] \neq 0.$$

3.4.1 Preliminary remarks

Lemma 19 *The following holds:*

$$[L_{MF}(u), L_{MF}(z)] = [\mathcal{K}, \Phi](\frac{1}{u} - \frac{1}{z}) = [\mathcal{K}, L_{MF}(z)](\frac{z}{u} - 1) \tag{72}$$

Remark 5

$$L_{MF}(u) = \frac{z}{u} L_{MF}(z) - K(\frac{z}{u} - 1) \qquad L_{MF}(z) = K(1 - \frac{u}{z}) + \frac{u}{z} L_{MF}(u) \tag{73}$$

Let A be a linear operator in $V \otimes V$. Let us denote $Tr_1 A$ the trace taken only over the first component, i.e. $(Tr_1 A)_{ij} = \sum_k A_{kk,ij}$.
By a straightforward calculation one has

Lemma 20

$$Tr_1(\overset{1}{A} P) = Tr_1(\overset{2}{A} P) = A \qquad Tr_1(\overset{1}{A}\overset{2}{B} P) = \sum_{ij} \sum_k A_{kj} B_{ik} e_{ij}$$

3.4.2 Quadratic Hamiltonians

Lemma 21 *The following holds:*

$$[TrL^n_{MF}(z), TrL^2_{MF}(u)] = Tr[L_{MF}(u), L^n_{MF}(z)] = 0 \tag{74}$$

Proof

$$\begin{aligned}
& [TrL^n_{MF}(z), TrL^2_{MF}(u)] = \text{ according to lemma 15} \\
= \;& 2Tr[L_{MF}(u), L^n_{MF}(z)]\frac{1}{z-u} = Tr\frac{2}{z-u}\sum_{i=0}^{n-1} L^i_{MF}(z)[L_{MF}(u), L_{MF}(z)]L^{n-i-1}_{MF}(z) \\
= \;& Tr\frac{2}{z-u}\sum_{i=0}^{n-1} L^i_{MF}(z)[\mathcal{K}, L_{MF}(z)](\frac{z}{u}-1)L^{n-i-1}_{MF}(z) \\
= \;& \frac{2}{z-u}(\frac{z}{u}-1)Tr[\mathcal{K}, L^n_{MF}(z)] = 0 \text{ because } \mathcal{K} \text{ is a } \mathbb{C}\text{-valued matrix}
\end{aligned} \tag{75}$$

□

3.4.3 Cartan subalgebra

We will prove that Cartan subalgebra is contained is the subalgebra in $U(\mathfrak{gl}(n))$ generated by the $Tr(L(z)^k)$ and Cartan subalgebra commutes with any element of type $Tr(L(z)^k)$.

Lemma 22 *Let us assume that $\mathcal{K} = diag\{k_j\}$, then the Cartan subalgebra generated by e_{jj} is contained in the subalgebra generated by coefficients in z of $TrL_{MF}(z)^k$.*
Proof Let us consider the residue of $TrL_{MF}(z)^k$ at $z = 0$

$$Res_{z=0}TrL^k_{MF}(z) = Res_{z=0}Tr\left(\mathcal{K} + \frac{\Phi}{z}\right)^k = kTr\mathcal{K}^{k-1}\Phi = \sum_j k_j^{k-1}e_{jj}.$$

The general choice of $\mathcal{K}$ imply that all the k_j are different. In this case the Vandermonde matrix $\{k_j^l\}$ is not degenerate and by taking linear combinations of the residues above one recovers e_{jj}.

Lemma 23 $\forall n, i$ *one has*

$$[TrL^n_{MF}(z), (L_{MF}(u))_{ii}] = [TrL^n_{MF}(z), e_{ii}] = 0.$$

Proof

$$\begin{aligned}
& [TrL^n_{MF}(z), (L_{MF}(u))_{ii}] = \sum_j [\overset{1}{L}{}^n_{MF}(z), \overset{2}{(}L_{MF}(u))]_{jj,ii} \\
= \;& \Big(Tr_1[\overset{1}{L}{}^n_{MF}(z), \overset{2}{(}L_{MF}(u))]\Big)_{ii} = \text{ /by the formula 44/} \\
= \;& \Big(Tr_1\Big((\overset{2}{L}{}^n_{MF}(z) - \overset{1}{L}{}^n_{MF}(z))\frac{P}{z-u} + \sum_{k=0}^{n-1}\overset{1}{L}{}^k_{MF}(z)(\overset{1}{L}_{MF}(u) - \overset{2}{L}_{MF}(u))\overset{2}{L}{}^{n-1-k}_{MF}(z)\frac{P}{z-u}\Big)\Big)_{ii}
\end{aligned}$$

$$
\begin{aligned}
&= \quad \text{/by lemma 20 /} = (L^n_{MF}(z)) - (L^n_{MF}(z))_{ii}\frac{1}{z-u} \\
&+ \quad \frac{1}{z-u}\sum_{k=0}^{n-1}\sum_j (L^k_{MF}(z)L_{MF}(u))_{ji}(L^{n-1-k}_{MF}(z))_{ij} - (L^k_{MF}(z))_{ji}(L_{MF}(u)L^{n-1-k}_{MF}(z))_{ij} \\
&= \quad \frac{1}{z-u}\sum_{k=0}^{n-1}\sum_j (L^k_{MF}(z)(\frac{z}{u}L_{MF}(z) - \mathcal{K}(\frac{z}{u}-1)))_{ji}(L^{n-1-k}_{MF}(z))_{ij} \\
&- \quad (L^k_{MF}(z))_{ji}(\frac{z}{u}L_{MF}(z) - \mathcal{K}(\frac{z}{u}-1)L^{n-1-k}_{MF}(z))_{ij} \\
&= \quad \frac{1}{z-u}\frac{z}{u}\sum_{k=0}^{n-1}\sum_j (L^k_{MF}(z)L_{MF}(z))_{ji}(L^{n-1-k}_{MF}(z))_{ij} \\
&- \quad (L^k_{MF}(z))_{ji}(L_{MF}(z)L^{n-1-k}_{MF}(z))_{ij} - \frac{1}{z-u}(\frac{z}{u}-1)\sum_{k=0}^{n-1}\sum_j (L^k_{MF}(z)\mathcal{K})_{ji}(L^{n-1-k}_{MF}(z))_{ij} \\
&- \quad (L^k_{MF}(z))_{ji}(\mathcal{K}L^{n-1-k}_{MF}(z))_{ij} = \frac{1}{z-u}\frac{z}{u}\sum_j (L^n_{MF}(z))_{ji}\delta_{ij} - \delta_{ij}(L^n_{MF}(z))_{ij} \\
&- \quad \frac{1}{z-u}(\frac{z}{u}-1)\sum_{k=0}^{n-1}\sum_j (L^k_{MF}(z)_{ji}k_iL^{n-1-k}_{MF}(z)_{ij} - L^k_{MF}(z)_{ji}k_iL^{n-1-k}_{MF}(z)_{ij}) = 0
\end{aligned}
$$

□

3.4.4 Commutativity of traces $[TrL^n_{MF}(z), TrL^3_{MF}(u)] = 0$ for n=1,...,5

Lemma 24

$$
[TrL^n_{MF}(z), TrL^3_{MF}(u)] = \frac{3}{u^2}\sum_{i=3}^{n-2}([TrL^i_{MF}(z), Tr\mathcal{K}^2L^{n-1-i}_{MF}(z)]) \tag{76}
$$

Corollary 10

$$
[TrL^n_{MF}(z), TrL^3_{MF}(u)] = 0, \ \textit{for n=1,...,5} \tag{77}
$$

Proof of the Corollary The corollary follows immediately for $n = 1, ..., 4$ from the lemma above - because summation in the formula 76 is out of range for such n. The only case is $n = 5$ then there is only one term in summation: $[TrL^3_{MF}(z), Tr\mathcal{K}^2L_{MF}(z)]$, this term equals to zero due to the lemma 23.
Proof of the lemma According to the lemma 21 one has $Tr[L^n_{MF}(z), L_{MF}(u)] = 0$. Hence using the lemma 17, we obtain

$$
[TrL^n_{MF}(z), TrL^3_{MF}(u)] = \frac{3}{z-u}Tr[L^2_{MF}(u), L^n_{MF}(z)] \tag{78}
$$

Let us simplify this expression:

$$
\begin{aligned}
& Tr[L_{MF}^n(z), L_{MF}^2(u)] = Tr[L_{MF}^n(z), (\frac{z}{u}L_{MF}(z) - K(\frac{z}{u}-1))^2] \\
= & -(\frac{z}{u}-1)\frac{z}{u}Tr[L_{MF}^n(z), (KL_{MF}(z) + L_{MF}(z)K)] \\
= & -(\frac{z}{u}-1)\frac{z}{u}TrL_{MF}^n(z)KL_{MF}(z) - L_{MF}(z)KL_{MF}^n(z)
\end{aligned}
$$

So we came to the expression which is considered in the lemma 27. Let us denote

$$B = L_{MF}^n(z), \qquad A = L_{MF}(z).$$

Then using

$$[\overset{1}{L}_{MF}(z), \overset{2}{L}_{MF}(z)] = \frac{1}{z}[\overset{1}{L}_{MF}(z) - \overset{1}{\mathcal{K}}, P] = (\overset{1}{L}_{MF}(z) - \overset{1}{\mathcal{K}} - \overset{2}{L}_{MF}(z) + \overset{2}{\mathcal{K}})P$$

we obtain

$$
\begin{aligned}
[\overset{1}{B}, \overset{2}{A}] &= [\overset{1}{L}{}_{MF}^n(z), \overset{2}{L}_{MF}(z)] = \\
&= \sum_{i=0}^{n-1} \frac{1}{z}\overset{1}{L}{}_{MF}^i(z)(\overset{1}{L}_{MF}(z) - \overset{1}{\mathcal{K}} - \overset{2}{L}_{MF}(z) + \overset{2}{\mathcal{K}})\overset{2}{L}{}_{MF}^{n-1-i}(z)P = \\
&= \frac{1}{z}(\overset{1}{L}{}_{MF}^n(z) - \overset{2}{L}{}_{MF}^n(z))P + \sum_{i=0}^{n-1}\frac{1}{z}\overset{1}{L}{}_{MF}^i(z)(-\overset{1}{\mathcal{K}} + \overset{2}{\mathcal{K}})\overset{2}{L}{}_{MF}^{n-1-i}(z)P. \qquad (79)
\end{aligned}
$$

As the result we get

$$
\begin{aligned}
& Tr(P\overset{2}{\mathcal{K}} + \overset{2}{\mathcal{K}}P)([\overset{1}{B}, \overset{2}{A}]) = Tr(P\overset{2}{K} + \overset{2}{K}P)[\overset{1}{L}{}_{MF}^n(z), \overset{2}{L}_{MF}(z)] \\
= & Tr(P\overset{2}{K} + \overset{2}{K}P)(\frac{1}{z}(\overset{1}{L}{}_{MF}^n(z) - \overset{2}{L}{}_{MF}^n(z))P + \sum_{i=0}^{n-1}\frac{1}{z}\overset{1}{L}{}_{MF}^i(z)(-\overset{1}{\mathcal{K}} + \overset{2}{\mathcal{K}})\overset{2}{L}{}_{MF}^{n-1-i}(z)P) \\
= & \frac{1}{z}Tr(\overset{2}{K}\overset{1}{L}{}_{MF}^n(z) + \overset{1}{\mathcal{K}}\overset{1}{L}{}_{MF}^n(z) - \overset{2}{\mathcal{K}}\overset{2}{L}{}_{MF}^n(z) - \overset{1}{\mathcal{K}}\overset{2}{L}{}_{MF}^n(z)) \\
+ & \frac{1}{z}Tr(\overset{2}{K} + \overset{1}{K})\sum_{i=0}^{n-1}\overset{1}{L}{}_{MF}^i(z)(-\overset{1}{\mathcal{K}} + \overset{2}{\mathcal{K}})\overset{2}{L}{}_{MF}^{n-1-i}(z) \\
= & \frac{1}{z}Tr(\overset{2}{K}\overset{1}{L}{}_{MF}^n(z) + \overset{1}{\mathcal{K}}\overset{1}{L}{}_{MF}^n(z) - \overset{2}{\mathcal{K}}\overset{2}{L}{}_{MF}^n(z) - \overset{1}{\mathcal{K}}\overset{2}{L}{}_{MF}^n(z)) \\
+ & \frac{1}{z}Tr(\overset{2}{K} + \overset{1}{K})\sum_{i=0}^{n-1}\overset{1}{L}{}_{MF}^i(z)(-\overset{1}{\mathcal{K}} + \overset{2}{\mathcal{K}})\overset{2}{L}{}_{MF}^{n-1-i}(z) \\
= & \frac{1}{z}\sum_{i=0}^{n-1}Tr(-\overset{2}{\mathcal{K}}\overset{1}{L}{}_{MF}^i(z)\overset{1}{\mathcal{K}}\overset{2}{L}{}_{MF}^{n-1-i}(z) +
\end{aligned}
$$

$$
\begin{aligned}
+ &\quad \overset{2}{\mathcal{K}}\overset{1}{L}{}^i_{MF}(z)\overset{2}{\mathcal{K}}\overset{2}{L}{}^{n-1-i}_{MF}(z) - \overset{1}{\mathcal{K}}\overset{1}{L}{}^i_{MF}(z)\overset{1}{\mathcal{K}}\overset{2}{L}{}^{n-1-i}_{MF}(z) + \overset{1}{\mathcal{K}}\overset{1}{L}{}^i_{MF}(z)\overset{2}{\mathcal{K}}\overset{2}{L}{}^{n-1-i}_{MF}(z)) \\
= &\quad \frac{1}{z}\sum_{i=0}^{n-1}(-TrL^i_{MF}(z)\mathcal{K}TrL^{n-1-i}_{MF}(z)\mathcal{K} \\
+ &\quad TrL^i_{MF}(z)Tr\mathcal{K}L^{n-1-i}_{MF}(z)\mathcal{K} - Tr\mathcal{K}L^i_{MF}(z)\mathcal{K}TrL^{n-1-i}_{MF}(z) + Tr\mathcal{K}L^i_{MF}(z)Tr\mathcal{K}L^{n-1-i}_{MF}(z)) \\
= &\quad \frac{1}{z}\sum_{i=0}^{n-1}(TrL^i_{MF}(z)Tr\mathcal{K}L^{n-1-i}_{MF}(z)\mathcal{K} - Tr\mathcal{K}L^i_{MF}(z)\mathcal{K}TrL^{n-1-i}_{MF}(z)) \\
= &\quad \frac{1}{z}\sum_{i=0}^{n-1}([TrL^i_{MF}(z), Tr\mathcal{K}L^{n-1-i}_{MF}(z)\mathcal{K}]) = /\text{using lemma 28 }/ \\
= &\quad \frac{1}{z}\sum_{i=3}^{n-2}([TrL^i_{MF}(z), Tr\mathcal{K}L^{n-1-i}_{MF}(z)\mathcal{K}])
\end{aligned}
$$

□

3.4.5 NONcommutativity of traces of L^6 and L^3

Lemma 25 $[TrL^6_{MF}(z), TrL^3_{MF}(z)] \neq 0$
Proof According to lemma 24 one has

$$
[TrL^6_{MF}(z), TrL^3_{MF}(z)] = \frac{3}{z^2}([TrL^3_{MF}(z), Tr\mathcal{K}^2L^2_{MF}(z)]) + \frac{3}{z^2}([TrL^4_{MF}(z), Tr\mathcal{K}^2L_{MF}(z)]) \tag{80}
$$

The second term equals to zero due to the lemma 23. Let us show that the first term is non zero.

$$
\begin{aligned}
&\quad [TrL^3_{MF}(z), Tr\mathcal{K}^2L^2_{MF}(z)] \\
= &\quad [Tr\frac{\mathcal{K}^2\Phi}{z} + \frac{2\mathcal{K}\Phi^2 + \Phi\mathcal{K}\Phi}{z^2}, 2Tr\frac{\mathcal{K}^3\Phi}{z}] + [Tr\frac{\mathcal{K}^2\Phi}{z} + \frac{2\mathcal{K}\Phi^2 + \Phi\mathcal{K}\Phi}{z^2}, Tr\frac{\mathcal{K}^2\Phi^2}{z^2}] \\
&\quad /\text{ according to lemmas 32, 29 the first commutator equals to zero,} \\
&\quad [Tr\tfrac{\mathcal{K}^2\Phi}{z}, Tr\tfrac{\mathcal{K}^2\Phi^2}{z^2}] = 0 \text{ by lemma 32/} \\
= &\quad \frac{1}{z^4}[Tr2\mathcal{K}\Phi^2 + \Phi\mathcal{K}\Phi, Tr\mathcal{K}^2\Phi^2] = \frac{1}{z^4}Tr[2\overset{1}{\mathcal{K}}\overset{1}{\Phi}{}^2 + \overset{1}{\Phi}\overset{1}{\mathcal{K}}\overset{1}{\Phi}, \overset{2}{\mathcal{K}}{}^2\overset{2}{\Phi}{}^2] \\
= &\quad \frac{1}{z^4}(2Tr\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2[\overset{1}{\Phi}{}^2, \overset{2}{\Phi}{}^2] + Tr(\overset{2}{\mathcal{K}}{}^2[\overset{1}{\Phi}, \overset{2}{\Phi}{}^2]\overset{1}{\mathcal{K}}\overset{1}{\Phi} + \overset{1}{\Phi}\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2[\overset{1}{\Phi}, \overset{2}{\Phi}{}^2])) \\
= &\quad \frac{1}{z^4}(2Tr\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2(P\overset{2}{\Phi}{}^3 - P\overset{1}{\Phi}{}^3 + P\overset{1}{\Phi}\overset{2}{\Phi}{}^2 - P\overset{1}{\Phi}{}^2\overset{2}{\Phi}) + Tr(\overset{2}{\mathcal{K}}{}^2[P, \overset{2}{\Phi}{}^2]\overset{1}{\mathcal{K}}\overset{1}{\Phi} + \overset{1}{\Phi}\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2[P, \overset{2}{\Phi}{}^2])) \\
= &\quad \frac{1}{z^4}(2(Tr\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2\overset{1}{\Phi}{}^3P - Tr\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2\overset{2}{\Phi}{}^3P + Tr\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2\overset{2}{\Phi}\overset{1}{\Phi}{}^2P - Tr\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2\overset{2}{\Phi}{}^2\overset{1}{\Phi}P) \\
+ &\quad Tr(\overset{2}{\mathcal{K}}{}^2P\overset{2}{\Phi}{}^2\overset{1}{\mathcal{K}}\overset{1}{\Phi} - \overset{2}{\mathcal{K}}{}^2\overset{2}{\Phi}{}^2P\overset{1}{\mathcal{K}}\overset{1}{\Phi} + \overset{1}{\Phi}\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2P\overset{2}{\Phi}{}^2 - \overset{1}{\Phi}\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}{}^2\overset{2}{\Phi}{}^2P)) \\
= &\quad \frac{1}{z^4}(2(Tr\mathcal{K}^3\Phi^3 - Tr\mathcal{K}^3\Phi^3 + Tr\mathcal{K}^2\Phi\mathcal{K}\Phi^2 - Tr\mathcal{K}^2\Phi^2\mathcal{K}\Phi)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad Tr(\Phi^2\mathcal{K}^3\Phi - \mathcal{K}^2\Phi^2\mathcal{K}\Phi + \Phi\mathcal{K}\Phi^2\mathcal{K}^2 - \Phi\mathcal{K}^3\Phi^2))\\
&= \quad \frac{1}{z^4}(2(Tr\mathcal{K}^2\Phi\mathcal{K}\Phi^2 - Tr\mathcal{K}^2\Phi^2\mathcal{K}\Phi) + Tr(-\mathcal{K}^2\Phi^2\mathcal{K}\Phi + \Phi\mathcal{K}\Phi^2\mathcal{K}^2))\\
&= \quad \frac{1}{z^4}3(Tr\mathcal{K}^2\Phi\mathcal{K}\Phi^2 - Tr\mathcal{K}^2\Phi^2\mathcal{K}\Phi)
\end{aligned}
$$

We have used $Tr\Phi^n\mathcal{K}\Phi - \Phi\mathcal{K}\Phi^n = 0$ for $n = 3$. This will be proved latter (see lemma 30). So we have just obtained that

$$[TrL^3(z), Tr\mathcal{K}^2L^2(z)] = \frac{3}{z^4}(Tr\mathcal{K}^2\Phi\mathcal{K}\Phi^2 - Tr\mathcal{K}^2\Phi^2\mathcal{K}\Phi)$$

and

$$[Tr\mathcal{K}\Phi^2, Tr\mathcal{K}^2\Phi^2] = [Tr\Phi\mathcal{K}\Phi, Tr\mathcal{K}^2\Phi^2] = Tr\mathcal{K}^2\Phi\mathcal{K}\Phi^2 - Tr\mathcal{K}^2\Phi^2\mathcal{K}\Phi.$$

Let us show that $Tr\mathcal{K}^2\Phi\mathcal{K}\Phi^2 - Tr\mathcal{K}^2\Phi^2\mathcal{K}\Phi \neq 0$.
Recall that we are considering the diagonal matrix $\mathcal{K} = \{k_j\}$, then:

$$
\begin{aligned}
Tr\mathcal{K}^2\Phi\mathcal{K}\Phi^2 - Tr\mathcal{K}^2\Phi^2\mathcal{K}\Phi &= \sum_{j,p} k_j^2\Phi_{jp}k_p(\Phi)^2_{pj} - k_j^2(\Phi)^2_{jp}k_p(\Phi)_{pj}\\
&= \sum_{j\neq p} k_j^2k_p(\Phi_{jp}(\Phi)^2_{pj} - (\Phi)^2_{jp}(\Phi)_{pj}).
\end{aligned}
$$

For this expression to be zero it should be that

$$\forall j, p : \Phi_{jp}(\Phi)^2_{pj} - (\Phi)^2_{jp}\Phi_{pj} = 0,$$

but it's not true for the case of $\mathfrak{gl}(3)$ and higher rank algebras, by the PBW property, due to the following

$$\Phi_{jp}(\Phi)^2_{pj} - (\Phi)^2_{jp}\Phi_{pj} = \Phi_{jp}\sum_l \Phi_{pl}\Phi_{lj} - \sum_l \Phi_{jl}\Phi_{lp}\Phi_{pj}.$$

Hence the term $\sum_l \Phi_{j,p}\Phi_{p,l}\Phi_{l,j}$ contains elements $\Phi_{p,l}$ which are not contained in the term: $-\sum_l \Phi_{j,l}\Phi_{l,p}\Phi_{p,j}$. $\square$

Remark 6 For the case of $\mathfrak{gl}(2)$ this is zero (we should check only j=1,p=2):

$$
\begin{aligned}
&\Phi_{12}\Phi_{21}\Phi_{11} + \Phi_{12}\Phi_{22}\Phi_{21} - \Phi_{11}\Phi_{12}\Phi_{21} - \Phi_{12}\Phi_{22}\Phi_{21} = [\Phi_{12}\Phi_{21}, \Phi_{11}]\\
&\quad = [\Phi_{12}, \Phi_{11}]\Phi_{21} + \Phi_{12}[\Phi_{21}, \Phi_{11}] = -\Phi_{12}\Phi_{21} + \Phi_{12}\Phi_{21} = 0
\end{aligned}
$$

3.4.6 Commutativity $[TrL^4_{MF}(z), TrL^4_{MF}(u)] = 0$

Lemma 26 *The following is true:*

$$[TrL^4_{MF}(z), TrL^4_{MF}(u)] = 0. \tag{81}$$

Combined with the results of the previous sections this lemma proves the following

Theorem 1 *Coefficients at $\frac{1}{z^i}$ of $TrL^l_{MF}(z)$, $l \leq N$ freely generates maximal commutative subalgebra in $U(\mathfrak{gl}(N))$, $N \leq 4$.*
Let us explain that the maximality and free generation follows from the results of Mishchenko and Fomenko [11], who proved that the coefficients at $\frac{1}{z}$ of $\mathsf{symb}(TrL^k_{MF}(z)) \in S(\mathfrak{gl}(n))$ generate maximal Poisson-commutative subalgebra in $S(\mathfrak{gl}(n))$ (for general $\mathcal{K}$).

Remark 7 According to the previous results the theorem above cannot be true for $N \geq 6$. We hope that it is still true for $N = 5$.
Proof of the lemma 26
We do not have the general formula for the commutators $[L^n_{MF}(z), L^4_{MF}(u)]$ and propose here a straightforward proof.

$$\begin{aligned}
& [TrL^4_{MF}(z), TrL^4_{MF}(u)] = [Tr(\mathcal{K} + \frac{\Phi}{z})^4, Tr(\mathcal{K} + \frac{\Phi}{u})^4] \\
= \ & [Tr(3\mathcal{K}^2\Phi^2 + 2\mathcal{K}\Phi\mathcal{K}\Phi + \Phi\mathcal{K}^2\Phi), 2Tr(\mathcal{K}\Phi^3 + \Phi\mathcal{K}\Phi^2)](\frac{1}{z^2u^3} - \frac{1}{z^3u^2}) \\
= \ & [Tr(4\mathcal{K}^2\Phi^2 + 2\mathcal{K}\Phi\mathcal{K}\Phi + [\Phi, \mathcal{K}^2\Phi]), 2Tr(2\mathcal{K}\Phi^3 + [\Phi, \mathcal{K}\Phi^2])](\frac{1}{z^2u^3} - \frac{1}{z^3u^2}) \\
= \ & / \text{ by lemma 33 } / = [Tr([\Phi, \mathcal{K}^2\Phi]), 2Tr(2\mathcal{K}\Phi^3 + [\Phi, \mathcal{K}\Phi^2])](\frac{1}{z^2u^3} - \frac{1}{z^3u^2}) \\
+ \ & [Tr(4\mathcal{K}^2\Phi^2 + 2\mathcal{K}\Phi\mathcal{K}\Phi), 2Tr([\Phi, \mathcal{K}\Phi^2])](\frac{1}{z^2u^3} - \frac{1}{z^3u^2}) \\
= \ & / \text{ by lemma 31 and the fact that } Tr\Phi^k \text{ are Casimirs } / \\
= \ & [-Tr(\mathcal{K}^2\Phi) \ \ TrId, 2Tr(2\mathcal{K}\Phi^3) - Tr(\mathcal{K}\Phi^2) \ \ TrId](\frac{1}{z^2u^3} - \frac{1}{z^3u^2}) \\
+ \ & [Tr(4\mathcal{K}^2\Phi^2 + 2\mathcal{K}\Phi\mathcal{K}\Phi), -2Tr(\mathcal{K}\Phi^2) \ \ TrId](\frac{1}{z^2u^3} - \frac{1}{z^3u^2}) \\
& / \text{ which is zero due to the lemma 33 and 32. } /
\end{aligned}$$

□

Analogously we can prove that

$$\begin{aligned}
[Tr(2\mathcal{K}^2\Phi^2 + \mathcal{K}\Phi\mathcal{K}\Phi), Tr(\Phi^l\mathcal{K}\Phi^n)] = 0 \\
[Tr(3\mathcal{K}^2\Phi^2 + 2\mathcal{K}\Phi\mathcal{K}\Phi + \Phi\mathcal{K}^2\Phi), Tr(\Phi^l\mathcal{K}\Phi^n)] = 0
\end{aligned} \tag{82}$$

hence the coefficient at $\frac{1}{z^2}$ of $TrL^4_{MF}(z)$ commutes with the coefficient at $\frac{1}{z^{l+n}}$ of $TrL^{l+n+1}_{MF}(z)$.

3.4.7 Auxiliary lemmas.

In this subsection we will prove some lemmas, which has been used in the calculations above, but may also represent an independent interest.

Lemma 27

$$TrB\mathcal{K}A - A\mathcal{K}B = TrP(\overset{2}{K} + \overset{1}{K})([\overset{1}{B}, \overset{2}{A}]) \text{ for } A, B : [A, B] = 0 \tag{83}$$

Proof Let us assume that $\mathcal{K} = diag\{k_j\}$. Then

$$
\begin{aligned}
&TrB\mathcal{K}A - A\mathcal{K}B = \sum_j k_j(\sum_i B_{ij}A_{ji} - A_{ij}B_{ji}) \\
= \;& \sum_j k_j(\sum_i A_{ji}B_{ij} - B_{ji}A_{ij} + [B_{ij}, A_{ji}] - [A_{ij}, B_{ji}]) \\
= \;& \text{ / using } [A,B]=0 \text{ / } = \sum_j k_j(\sum_i [B_{ij}, A_{ji}] - [A_{ij}, B_{ji}]) \\
= \;& \sum_j k_j(\sum_i [B_{ij}, A_{ji}] - [A_{ij}, B_{ji}]) = \sum_j k_j(\sum_i [\overset{1}{B}, \overset{2}{A}]_{ij,ji} - [\overset{1}{A}, \overset{2}{B}]_{ij,ji}) = \\
= \;& \sum_j \sum_i k_j(([\overset{1}{B}, \overset{2}{A}]P)_{ii,jj} - ([\overset{1}{A}, \overset{2}{B}]P)_{ii,jj}) = Tr\overset{2}{K}(([\overset{1}{B}, \overset{2}{A}]P) - ([\overset{1}{A}, \overset{2}{B}]P)) = \\
= \;& Tr\overset{2}{K}(([\overset{1}{B}, \overset{2}{A}]P) - (P[\overset{2}{A}, \overset{1}{B}])) = Tr\overset{2}{K}(([\overset{1}{B}, \overset{2}{A}]P) + (P[\overset{1}{B}, \overset{2}{A}])) \\
= \;& TrP(\overset{2}{K} + \overset{1}{K})([\overset{1}{B}, \overset{2}{A}]).
\end{aligned}
$$

□

Lemma 28

$$
[TrL^2_{MF}(u), Tr\mathcal{K}^2 L^n_{MF}(z)] = \frac{2}{u}[Tr\mathcal{K}\Phi, Tr\mathcal{K}^2 L^n_{MF}(z)] = 0 \tag{84}
$$

Proof Let us note that

$$
TrL^2_{MF}(u) = Tr\mathcal{K}^2 + \frac{2}{u}Tr\mathcal{K}\Phi + \frac{1}{u^2}Tr\Phi^2.
$$

Recalling that $Tr\mathcal{K}^2, Tr\Phi^2$ commute with everything we see that it is enough to prove

$$
[Tr\mathcal{K}\Phi, Tr\mathcal{K}^2 L^n_{MF}(z)] = 0
$$

or equivalently

$$
[Tr\mathcal{K}L_{MF}(u), Tr\mathcal{K}^2 L^n_{MF}(z)] = 0.
$$

So let us calculate:

$$
\begin{aligned}
&[Tr\mathcal{K}^2 L^n_{MF}(z), Tr\mathcal{K}L_{MF}(u)] = Tr[\overset{1}{\mathcal{K}}{}^2 \overset{1}{L}{}^n_{MF}(z)), \overset{2}{K}\overset{2}{L}_{MF}(u)] \\
= \;& Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}[\overset{1}{L}{}^n_{MF}(z), \overset{2}{L}_{MF}(u)] = \text{ / by lemma 10 and formula 44 / } \\
= \;& Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}(\overset{2}{L}{}^n(z) - \overset{1}{L}{}^n(z))\frac{P}{z-u} + Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}\sum_{i=0}^{n-1}\overset{1}{L}{}^i(z)(\overset{1}{L}(u) - \overset{2}{L}(u))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u} \\
= \;& Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}\sum_{i=0}^{n-1}\overset{1}{L}{}^i(z)(\frac{z}{u}\overset{1}{L}(z) - \overset{1}{\mathcal{K}}(\frac{z}{u}-1) - \frac{z}{u}\overset{2}{L}(z) + \overset{2}{\mathcal{K}}(\frac{z}{u}-1))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u}
\end{aligned}
$$

$$
\begin{aligned}
&= Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}\sum_{i=0}^{n-1}\overset{1}{L}{}^i(z)(\frac{z}{u}\overset{1}{L}(z)-\frac{z}{u}\overset{2}{L}(z))\overset{2}{L}{}^{n-1-i}(z)\frac{P}{z-u} \\
&= \frac{z}{u}Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}(\overset{1}{L}{}^n(z)\frac{P}{z-u}-\overset{2}{L}{}^n(z)\frac{P}{z-u}) \\
&= \frac{z}{u}(Tr\mathcal{K}^2L^n(z)\mathcal{K}-Tr\mathcal{K}^2\mathcal{K}L^n(z))\frac{1}{z-u})=0
\end{aligned}
$$

□

Lemma 29

$$[Tr\Phi A\Phi, TrB\Phi]=0, \quad for \;\; A,B:[A,B]=0$$

Proof By a straightforward calculation.

Lemma 30

$$Tr\Phi^n\mathcal{K}\Phi-\Phi\mathcal{K}\Phi^n=0.$$

Proof According to the formula 83:

$$
\begin{aligned}
& Tr\Phi^n\mathcal{K}\Phi-\Phi\mathcal{K}\Phi^n = TrP(\overset{2}{\mathcal{K}}+\overset{1}{\mathcal{K}})([\overset{1}{\Phi}{}^n,\overset{2}{\Phi}]) = TrP(\overset{2}{\mathcal{K}}+\overset{1}{\mathcal{K}})([\overset{1}{\Phi}{}^n,P]) \\
= \;& TrP(\overset{2}{\mathcal{K}}+\overset{1}{\mathcal{K}})(\overset{1}{\Phi}{}^nP-\overset{2}{\Phi}{}^nP) = Tr(\overset{2}{\mathcal{K}}\overset{1}{\Phi}{}^n-\overset{2}{\mathcal{K}}\overset{2}{\Phi}{}^n)+(\overset{1}{\mathcal{K}}\overset{1}{\Phi}{}^n-\overset{1}{\mathcal{K}}\overset{2}{\Phi}{}^n) \\
= \;& Tr\mathcal{K}Tr\Phi^n-TrIdTr\mathcal{K}\Phi^n+Tr\mathcal{K}\Phi^nTrId-Tr\mathcal{K}Tr\Phi^n)=0
\end{aligned}
$$

□

Lemma 31

$$Tr[\Phi,\mathcal{K}^l\Phi^m]=Tr\mathcal{K}^l\;\;Tr\Phi^m-Tr\mathcal{K}^l\Phi^m\;\;TrId \tag{85}$$

Proof

$$
\begin{aligned}
Tr[\Phi,\mathcal{K}^l\Phi^m] &= Tr[\overset{1}{\Phi},\overset{2}{\mathcal{K}}{}^l\overset{2}{\Phi}{}^m]P = Tr\overset{2}{\mathcal{K}}{}^l[\overset{1}{\Phi},\overset{2}{\Phi}{}^m]P = Tr\overset{2}{\mathcal{K}}{}^l[P,\overset{2}{\Phi}{}^m]P = \\
&= Tr\overset{2}{\mathcal{K}}{}^l\overset{1}{\Phi}{}^m-Tr\overset{2}{\mathcal{K}}{}^l\overset{2}{\Phi}{}^m = Tr\mathcal{K}^l\;\;Tr\Phi^m-Tr\mathcal{K}^l\Phi^m\;\;TrId
\end{aligned} \tag{86}
$$

□

Lemma 32 (due to Skrypnyk [13]):

$$[TrA\Phi,TrB\Phi^n]=0 \quad \text{where} \quad [A,B]=0 \tag{87}$$

Proof:

$$
\begin{aligned}
Tr[\overset{1}{A}\overset{1}{\Phi},\overset{2}{B}\overset{2}{\Phi}{}^n] &= Tr\overset{1}{A}\overset{2}{B}[P,\overset{2}{\Phi}{}^n] \\
&= Tr\overset{1}{A}\overset{2}{B}\overset{1}{\Phi}{}^nP-Tr\overset{1}{A}\overset{2}{B}\overset{2}{\Phi}{}^nP = TrA\Phi^nB-TrAB\Phi^n=0
\end{aligned} \tag{88}
$$

□

Lemma 33

$$[Tr(2\mathcal{K}^2\Phi^2 + \mathcal{K}\Phi\mathcal{K}\Phi), Tr\mathcal{K}\Phi^n] = 0 \tag{89}$$

Proof

$$\begin{aligned}
& [Tr\mathcal{K}^2\Phi^2, Tr\mathcal{K}\Phi^n] = Tr[\overset{1}{\mathcal{K}}{}^2\overset{1}{\Phi}{}^2, \overset{2}{\mathcal{K}}\overset{2}{\Phi}{}^n] = Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}[\overset{1}{\Phi}{}^2, \overset{2}{\Phi}{}^n] \\
= \; & / \text{ by lemma 3 } / = Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}(\sum_{a=1,2} P\overset{1}{\Phi}{}^{a-1}\overset{2}{\Phi}{}^{n+2-a} - P\overset{1}{\Phi}{}^{n+2-a}\overset{2}{\Phi}{}^{a-1}) \\
= \; & Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}(P\overset{2}{\Phi}{}^{n+1} - P\overset{1}{\Phi}{}^{n+1} + P\overset{1}{\Phi}\overset{2}{\Phi}{}^n - P\overset{1}{\Phi}{}^n\overset{2}{\Phi}) = Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}P(\overset{1}{\Phi}\overset{2}{\Phi}{}^n - \overset{1}{\Phi}{}^n\overset{2}{\Phi}) \\
= \; & Tr\overset{2}{\mathcal{K}}\overset{2}{\Phi}\overset{1}{\mathcal{K}}{}^2\overset{1}{\Phi}{}^n P - Tr\overset{2}{\mathcal{K}}\overset{2}{\Phi}{}^n\overset{1}{\mathcal{K}}{}^2\overset{1}{\Phi}P = Tr\mathcal{K}\Phi\mathcal{K}^2\Phi^n - Tr\mathcal{K}\Phi^n\mathcal{K}^2\Phi
\end{aligned} \tag{90}$$

$$\begin{aligned}
& [Tr\mathcal{K}\Phi\mathcal{K}\Phi, Tr\mathcal{K}\Phi^n] = Tr\overset{1}{\mathcal{K}}\overset{1}{\Phi}\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}[\overset{1}{\Phi}, \overset{2}{\Phi}{}^n] + \overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}[\overset{1}{\Phi}, \overset{2}{\Phi}{}^n]\overset{1}{\mathcal{K}}\overset{1}{\Phi} \\
= \; & / \text{ by lemma 2 } / \\
= \; & Tr\overset{1}{\mathcal{K}}\overset{1}{\Phi}\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}\overset{1}{\Phi}{}^n P - \overset{1}{\mathcal{K}}\overset{1}{\Phi}\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}\overset{2}{\Phi}{}^n P + \overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}\overset{1}{\Phi}{}^n\overset{2}{\mathcal{K}}\overset{2}{\Phi}P\overset{1}{\mathcal{K}}\overset{2}{\mathcal{K}}\overset{2}{\Phi}{}^n\overset{2}{\mathcal{K}}\overset{2}{\Phi}P \\
= \; & Tr\mathcal{K}^2\Phi\mathcal{K}\Phi^n - \mathcal{K}\Phi\mathcal{K}^2\Phi^n + \mathcal{K}\Phi^n\mathcal{K}^2\Phi - \mathcal{K}^2\Phi^n\mathcal{K}\Phi \\
= \; & Tr[\mathcal{K}^2\Phi, \mathcal{K}\Phi^n] - 2\mathcal{K}\Phi\mathcal{K}^2\Phi^3 + 2\mathcal{K}\Phi^n\mathcal{K}^2\Phi - [\mathcal{K}^2\Phi^n, \mathcal{K}\Phi] \\
= \; & 2Tr\mathcal{K}\Phi^n\mathcal{K}^2\Phi - 2Tr\mathcal{K}\Phi\mathcal{K}^2\Phi^n + Tr[\mathcal{K}^2\Phi, \mathcal{K}\Phi^n] - Tr[\mathcal{K}^2\Phi^n, \mathcal{K}\Phi]
\end{aligned} \tag{91}$$

$$\begin{aligned}
Tr[\mathcal{K}^2\Phi, \mathcal{K}\Phi^n] &= Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}[P, \overset{2}{\Phi}{}^n]P = Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}\overset{1}{\Phi}{}^n - Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}\overset{2}{\Phi}{}^n \\
&= Tr\mathcal{K}^2\Phi^n Tr\mathcal{K} - Tr\mathcal{K}^2 Tr\mathcal{K}\Phi^n \qquad (92) \\
Tr[\mathcal{K}^2\Phi^n, \mathcal{K}\Phi] &= Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}[\overset{1}{\Phi}{}^n, P]P = Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}\overset{1}{\Phi}{}^n - Tr\overset{1}{\mathcal{K}}{}^2\overset{2}{\mathcal{K}}\overset{2}{\Phi}{}^n \\
&= Tr\mathcal{K}^2\Phi^n Tr\mathcal{K} - Tr\mathcal{K}^2 Tr\mathcal{K}\Phi^n \qquad (93)
\end{aligned}$$

Hence

$$Tr[\mathcal{K}^2\Phi, \mathcal{K}\Phi^n] = Tr[\mathcal{K}^2\Phi^n, \mathcal{K}\Phi] \tag{94}$$

$$[Tr\mathcal{K}\Phi\mathcal{K}\Phi, Tr\mathcal{K}\Phi^n] = 2Tr\mathcal{K}\Phi^n\mathcal{K}^2\Phi - 2Tr\mathcal{K}\Phi\mathcal{K}^2\Phi^n \tag{95}$$

$$[Tr(2\mathcal{K}^2\Phi^2 + \mathcal{K}\Phi\mathcal{K}\Phi), Tr\mathcal{K}\Phi^n] = 0 \tag{96}$$

The lemma is proved. $\square$

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